Self-assessment answers: 15 Further integration

1. $V = \int_0^{\pi} \pi y^2 \,\mathrm{d}x$

$= \pi \int_0^{\pi} \sin 3x \,\mathrm{d}x$

$= -\frac{\pi}{3}[\cos 3x]_0^{\pi}$

$= \frac{2\pi}{3}$ *[5 marks]*

2. $u = x - 1 \;\Rightarrow\; \mathrm{d}u = \mathrm{d}x$

$\int x^2 \sqrt{x-1}\,\mathrm{d}x = \int (u+1)^2 \sqrt{u}\,\mathrm{d}u$

$= \int u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\,\mathrm{d}u$

$= \frac{2}{7}u^{\frac{7}{2}} + \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + c$

$= \left(\frac{2}{7}(x-1)^2 + \frac{4}{5}(x-1) + \frac{2}{3}\right)(x-1)^{\frac{3}{2}} + c$ *[6 marks]*

3. Let $u = x^2 - x + 5$

$\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x - 1$

When $x = 3$, $u = 11$, and when $x = 1$, $u = 5$.

$\int_1^3 \frac{2x-1}{\left(x^2-x+5\right)^3}\,\mathrm{d}x = \int_5^{11} \frac{2x-1}{u^3}\frac{\mathrm{d}u}{2x-1}$

$= \int_5^{11} \frac{1}{u^3}\,\mathrm{d}u = -\frac{1}{2}\left[\frac{1}{u^2}\right]_5^{11}$

$= \frac{48}{3025}$ *[6 marks]*

4. (a) $a = \frac{dv}{dt} = 3e^{-2t}\cos t - 6e^{-2t}\sin t = 3e^{-2t}(\cos t - 2\sin t)$

When at maximum velocity, $a = 0 \Rightarrow \tan t = 0.5$ (on the first occasion, since the exponential means that it will never again reach that level).

$\Rightarrow t = 0.464$ s

(b) $a(3) = -0.00946 \text{ ms}^{-2}$

(c) v is positive for $0 \le t \le \pi$, so distance travelled equals displacement at $t = 3$.

distance $= \int_0^3 v\,dt = 0.601$ m (GDC)

(d) Displacement $x(t) = \int_0^t 3e^{-2u}\sin u\,du$

$$= \left[-\frac{3}{2}e^{-2u}\sin u\right]_0^t + \int_0^t \frac{3}{2}e^{-2u}\cos u\,du \qquad \text{(integration by parts)}$$

$$= -\frac{3}{2}e^{-2t}\sin t + \left(\left[-\frac{3}{4}e^{-2u}\cos u\right]_0^t - \int_0^t \frac{3}{4}e^{-2u}\sin u\,du\right) \qquad \text{(integration by parts)}$$

$$= -\frac{3}{2}e^{-2t}\sin t + \left(\frac{3}{4} - \frac{3}{4}e^{-2t}\cos t - \frac{1}{4}x(t)\right)$$

$$\Rightarrow \quad x(t) = \frac{4}{5}\left(\frac{3}{4} - \frac{3}{4}e^{-2t}\cos t - \frac{3}{2}e^{-2t}\sin t\right) = \frac{3}{5}(1 - e^{-2t}\cos t - 2e^{-2t}\sin t)$$

[13 marks]